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On some recent questions in equivariant simple homotopy theory

Sören Illman

The common theme of the different sections in this paper is the operation of restricting the transformation group in equivariant simple homotopy theory.

In the case of a finite transformation group G one does not encounter any fundamental geometric problems and one can directly go on to establish formulas for the restriction homomorphism between equivariant Whitehead groups. This is the topic of Section 1. In the case of finite transformation groups it is possible to give formulas for the restriction homomorphism between equivariant Whitehead groups in complete generality. We have, however, here limited ourselves to present the case where G is an arbitrary finite group but the G-CW complexes are assumed to be of the type where all fixed point sets are non-empty, connected and simply-connected.

In Sections 2-4 the transformation group G is an arbitrary compact Lie group. Of the results presented in these sections we in particular want to mention the following one. In Section 4 we obtain a class of equivariant triangulations of compact smooth G-manifolds which behaves well with respect to the operation of restricting the transformation group to any closed subgroup H of G. This result provides a solution of Problem 4.1 in [16].

We are in this paper concerned with equivariant simple-homotopy theory and the equivariant Whitehead group $Wh_G(X)$ as defined in [9]. This theory is valid for an arbitrary compact Lie group G, and for arbitrary G-CW complexes X. For a different approach, valid when G is a finite group and all fixed point sets are non-empty, connected and simply connected, we refer to Rothenberg [15]. For papers dealing with the algebraic determination of the equivariant Whitehead group we refer to [5], [1], [11] and [2]. In the present paper we only use the algebraic determination of $Wh_G(X)$ in a special case, in fact, this case is exactly the same as the one we mentioned above in connection with the paper [15] by Rothenberg.

My work on the problem considered in Section 1 was prompted by a question of Kawakubo in a colloqium talk by him at Osaka University in February 1987. I am very grateful to him for posing the question. The work described in Sections 3 and 4 is to a large extent inspired by the paper [14] of Matumoto and Shiota. It should here be pointed out that the work presented in Sections 2, 3 and 4 represents work in progress. In my talk at the Symposium, Theorem F in Section 4 was only given as a Conjecture. Except for this change we have tried to follow the actual lecture as closely as possible in writing this paper. For example, Theorems A, B, C, D and Corollary E are the same as the results given as A, B, C, D and E in the actual lecture.

All my work has greatly benefited from the excellent working conditions here at R.I.M.S., and from the very kind reception I have received both here in Kyoto and at the many other Universities in Japan that I have visited. For all this I wish to express my sincere gratitude.

Equivariant
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bring solution clear

1. The restriction homomorphism $\mathrm{Res}^G_H : \mathrm{Wh}_G(X) \longrightarrow \mathrm{Wh}_H(X)$ in the case when G is a finite group

In the case when G is a finite group, and $H<G$ is an arbitrary subgroup of G, there are no problems in obtaining a well-defined restriction homomorphism

$$(*) \qquad \mathrm{Res}^G_H : \mathrm{Wh}_G(X) \longrightarrow \mathrm{Wh}_H(X) .$$

Recall that an element of $\mathrm{Wh}_G(X)$ is an equivalence class

$$s_G(V,X) = s(V,X) ,$$

where V is any finite G-CW complex containing X as a strong G-deformation retract. The equivalence relation is defined as follows:

$$(V,X) \underset{G}{\sim} (V',X) \Leftrightarrow \text{there is a } G\text{-equivariant formal deformation rel } X \text{ from } V \text{ to } V' .$$

When G is a finite group a G-CW complex X consits of an ordinary CW complex X together with a simplicial action of G on X, such that if $g \in G$ fixes a point in an open cell $\overset{\circ}{c}$ of X then g fixes all of c pointwise. Thus, when restricting the transformation group G to the subgroup H we automatically obtain a finite H-CW complex X. Given a finite G-CW pair (V,X), as above, we obtain a finite H-CW pair (V,X), with X a strong H-deformation retract of V. Furthermore it is immediately seen that a G-equivariant formal deformation rel X induces an H-equivariant formal deformation rel X. Thus, given an arbitrary element $s_G(V,X) \in Wh_G(X)$ we obtain a well-defined unique element $s_H(V,X) \in Wh_H(X)$. The restriction homomorphism (*) is defined by

$$Res_H^G(s_G(V,X)) = s_H(V,X) .$$

We wish to give an algebraic description of the restriction homomorphism (*), when both $Wh_G(X)$ and $Wh_H(X)$ are given in their algebraic forms as direct sums of ordinary Whitehead groups of various discrete groups.

In order to simplify the situation and our notation we shall here assume that the G-CW complex X is such that each fixed point set X^Q, $Q<G$, is non-empty, connected and simply connected. In this case we have

$$Wh_G(X) \cong \bigoplus_{(Q)_G} Wh(W_GQ) ,$$

where the direct sum is over the set of all G-conjugacy classes

$(Q)_G$, of arbitrary subgroups of G . Here we have denoted

$$W_G Q = N_G Q/Q ,$$

where $N_G Q$ is the normalizer of Q in G . Analogously we have

$$Wh_H(X) \cong \bigoplus_{(R)_H} Wh(W_H R) ,$$

where the direct sum is over the set of all H-conjugacy classes $(R)_H$, of arbitrary subgroups of H , and we have denoted

$$W_H R = N_H R/R$$

with $N_H R$ denoting the normalizer of $R<H$ in H .

Our problem is to determine the question mark in the commutative diagram

$$\begin{array}{ccc} Wh_G(X) & \cong & \bigoplus_{(Q)_G} Wh(W_G Q) \\ \Big\downarrow{\scriptstyle Res^G_H} & & \Big\downarrow{\scriptstyle ?} \\ Wh_H(X) & \cong & \bigoplus_{(R)_H} Wh(W_H R) \end{array} .$$

As one can see from the algebraic description of $Wh_G(X)$, given above, the equivariant Whitehead group $Wh_G(X)$ is the same one for each G-CW complex X with the property that all fixed point sets X^Q , $Q<G$, are non-empty, connected and simply-connected. (For a general result on isomorphisms between equivariant Whitehead groups, see S. Araki [2, Theorem 9.3].) That is, in the special case that we are considering here the

actual geometry of the G-space X has no influence on $Wh_G(X)$, and $Wh_G(X)$ is in this case only depending on the group G. Thus, our problem to determine the question mark in the above diagram is the same one even if we take $X=\{*\}$. Our problem here, in the special case we are considering, is a purely combinatorial problem related to the subgroup H of the finite group G.

Given a subgroup Q of G, representing the G-conjugacy class $(Q)_G$, and a subgroup R of H, representing the H-conjugacy class $(R)_H$, we shall define a homomorphism

$$\rho(R,Q)\colon Wh(W_GQ) \longrightarrow Wh(W_HR)$$

such that the following holds:

If $Q' \in (Q)_G$ and $R' \in (R)_H$ are some other representatives for the G-conjugacy class $(Q)_G$ and the H-conjugacy class $(R)_H$, respectively, then the following diagram commutes

$$\begin{array}{ccc} Wh(W_GQ) & \xrightarrow{\rho(R,Q)} & Wh(W_HR) \\ {\scriptstyle \gamma(g)_*}\Big\downarrow\cong & & \cong\Big\downarrow{\scriptstyle \gamma(h)_*} \\ Wh(W_GQ') & \xrightarrow{\rho(R',Q')} & Wh(W_HR') \end{array} .$$

Here $\gamma(g)_*$ is the isomorphism induced by the isomorphism $\gamma(g)\colon W_GQ \longrightarrow W_GQ'$, defined by $\gamma(g)(nQ)=(gng^{-1})Q'$, where $n \in N_GQ$ and g is any element of G such that $gQg^{-1}=Q'$. The isomorphism $\gamma(g)_*$ is independent of the choice of the element $g \in G$, since an inner automorphism of a discrete group induces the identity map on the corresponding Whitehead group.

Thus $\gamma(g)_*$ gives us a canonical isomorphism from $Wh(W_GQ)$ to $Wh(W_GQ')$. The canonical isomorphism $\gamma(h)_*$ is defined analogously.

In order to give the definition of the homomorphism $\rho(R,Q)$ we proceed as follows. Consider the homogeneous space G/Q together with two actions on it; the right N_GQ-action given by

$$G/Q \times N_GQ \longrightarrow G/Q$$

$$(gQ,n) \longmapsto gnQ ,$$

and the standard action of H on the left.
These two actions on G/Q commute and we denote by

$$H \backslash (G/Q) / N_GQ$$

the total orbit space of the simultaneous left H-action and right N_GQ-action on G/Q . Observe that all points in an N_GQ-orbit in G/Q have the same H-isotropy subgroup. Let

$$[H \backslash (G/Q) / N_GQ]_{(R)_H} = \{H(g_1Q)N_GQ ,\dots, H(g_rQ)N_GQ\}$$

be the set of all total orbits which have H-isotropy type equal to $(R)_H$, and where we moreover have chosen the representing elements $g_1Q ,\dots, g_rQ \in G/Q$ such that they have H-isotropy subgroup equal to R . Thus we have

$$g_iQg_i^{-1} \cap H = R , \quad i=1,\dots,r .$$

It should be observed that for a given H-isotropy type, that is, an H-conjugacy class $(R)_H$, we may of course very well have

$$[H\backslash(G/Q)/N_GQ]_{(R)_H} = \phi ,$$

i.e., $r=0$. We denote

$$Q_i = g_iQg_i^{-1} , \quad i = 1,\dots,r ,$$

thus

$$Q_i \cap H = R , \; i = 1 ,\dots, r .$$

For each i, $1\leqq i\leqq r$, we define $\varrho_i : Wh(W_GQ) \to Wh(W_HR)$ by the following commutative diagram

$$\begin{array}{ccccc}
Wh(W_GQ) & & \xrightarrow{\quad\quad \rho_i \quad\quad} & & Wh(W_HR) \\
\gamma(g_i)_* \downarrow \cong & & & & \| \\
Wh(W_GQ_i) & \xrightarrow{res} & Wh((N_GQ_i\cap H)/(Q_i\cap H)) & \xrightarrow{ind} & Wh((N_GR\cap H)/R)
\end{array}$$

Here res and ind denote ordinary restriction and induction homomorphisms between Whitehead groups. Observe that $N_GQ_i \cap H < N_GR\cap H$.

We now set

$$\varrho(R,Q) = \sum_{i=1}^{r} \varrho_i : Wh(W_GQ) \longrightarrow Wh(W_HR) .$$

(In case $r=0$ the homomorphism $\varrho(R,Q)$ is the zero-homomorphism.) Then $\varrho(R,Q)$ is a well-defined homomorphism which is independent of the choice of the elements $g_1,\dots,g_r \in G$.

Theorem A. Let X be as above. Then the following diagram commutes

$$\begin{array}{ccc} Wh_G(X) & \cong & \bigoplus_{(Q)_G} Wh(W_GQ) \\ {\scriptstyle Res^G_H}\downarrow & & \downarrow {\scriptstyle \bar{\rho} = \oplus \rho(R,Q)} \\ Wh_H(X) & \cong & \bigoplus_{(R)_H} Wh(W_HR) \end{array} .$$

Here the notation $\oplus \rho(R,Q)$ has the obvious meaning; that is, $\bar{\rho}=\oplus \rho(R,Q)$ is the homomorphism (between the indicated direct sums) for which the induced homomorphism from the summand $Wh(W_GQ)$ to the summand $Wh(W_HR)$ equals $\rho(R,Q)$.

If G is abelian we have $r=1$ or 0 , and when $r=1$, i.e., when $Q \cap H=R$, we have that

$$\rho_1 = res^{G/Q}_{H/H\cap Q} : Wh(G/Q) \longrightarrow Wh(H/H\cap Q) = Wh(H/R) .$$

Thus, in the case when G is abelian (and X is as before) $\bar{\rho}$ is the direct sum of the ordinary restriction homomorphisms $res^{G/Q}_{H/H\cap Q}$, where Q runs through the set of all subgroups of G . This case, where G is abelian, is due to Dovermann-Rothenberg [3, Lemma 3.2].

2. <u>The restriction homomorphism</u> $\mathrm{Res}_H^G : \mathrm{Wh}_G(X) \to \mathrm{Wh}_H(X)$ <u>in the case when G is a compact Lie group</u>

In this section G denotes a compact Lie group and H is a fixed closed subgroup of G. Let X be a finite G-CW complex. We wish to define a restriction homomorphism

$$(**) \qquad \mathrm{Res}_H^G : \mathrm{Wh}_G(X) \longrightarrow \mathrm{Wh}_H(X) .$$

When G is a non-discrete compact Lie group (more precisely, when G/H is non-discrete) the task to obtain (**) as a well defined homomorphism requires that one first solves some geometric problems.

Given a finite G-CW complex X we are first of all faced with the problem of trying to give the H-space X the structure of a finite H-CW complex. If we manage to solve this first problem we are still left with the question of whether our solution gives rise to a well-defined simple H-homotopy type for X; that is, whether the obtained finite H-CW complex structure on X is unique up to a simple H-homotopy equivalence.

Before we continue the discussion let me point out the following.

<u>Remark.</u> One may also take the more general point of view that the equivariant Whitehead group $\mathrm{Wh}_G(X)$ is defined for an arbitrary G-space X, and the elements of $\mathrm{Wh}_G(X)$ are then the appropriate equivalence classes of G-pairs (V,X), where X is a strong G-deformation retract of V, and V is obtained from X by adjoining a finite number of G-equivariant cells.

When restricting the given action of G on X to an action of H on X we then have that X is an H-space and in this generalized context the H-equivariant Whitehead group $Wh_H(X)$ automatically has a well-defined meaning. It should however be observed that in order to have a well-defined homomorphism (**) one is still left with the same kind of problems as in the case, discussed above, when X is assumed to be a finite G-CW complex. One first of all has the problem of trying to show that the H-space V can be obtained from the H-space X by a finite number of H-equivariant cells, and furthermore one needs to show that this can be done in such a way that the obtained relative H-CW complex (V,X) determines a well-defined unique element in $Wh_H(X)$.

Thus we see that although the method of defining $Wh_G(X)$ for an arbitrary G-space X does have the advantage that the group $Wh_H(X)$ is automatically defined it does not however help us with the problems we encounter in trying to obtain a well-defined restriction homomorphism (**) . We may therefore as well return to discuss the case where X is assumed to be a finite G-CW complex.

Let us first note that a special case of the main theorem in [12] gives us the following result.

<u>Theorem B.</u> Let X be a finite G-CW complex, where G is a compact Lie group, and let H be a closed subgroup of G . Then there exists a <u>finite</u> H-CW complex $\check{X}$ of the same H-homotopy type as the H-space X .

It should be observed that it is not the H-space X itself that is given a finite H-CW complex structure. The finite

H-CW complex $\check{X}$ has the same H-homotopy type as the H-space X and $\check{X}$ is also in other aspects a good model for the H-space X (see [12] for more details), but no claim is made that $\check{X}$ is H-homeomorphic to X . In fact we gave in [12] an example which shows that in general the H-space X does not automatically inherit an H-CW complex structure from the given G-CW complex structure on X . This example does not rule out the possibility that the H-space X could be given an H-CW complex structure in some other way. But, in my opinion, the example in [12] clearly indicates that the search for H-CW complex structures on H-spaces X arising from G-CW complexes X , is not a very productive line of investigation.

In [12] we were not at all concerned with equivariant simple-homotopy theory, so no effort is made to make the construction such that the finite H-CW complex $\check{X}$ would be well-defined up to a simple H-homotopy equivalence. However, a similar technique as in [12], but with some important modifications, gives a proof of the following.

Theorem C. Let H be a closed subgroup of the compact Lie group G . Then one can associate to any finite G-CW complex X a finite H-CW complex $esh_H(X)$ and an H-homotopy equivalence $\mu: X \to esh_H(X)$ in such a way that the construction is well-defined up to a simple H-homotopy equivalence.

The last statement in Theorem C means that if we by some other choices in the construction arrive at the finite H-CW complex $esh_H(X)'$ and the H-homotopy equivalence $\mu': X \to esh_H(X)'$ then we have an H-homotopy commutative diagram

$$\begin{tikzcd} & X \arrow[dl, "\mu"'] \arrow[dr, "\mu'"] & \\ esh_H(X) \arrow[rr, "\sigma"] & & esh_H(X)' \end{tikzcd}$$

where σ is a simple H-homotopy equivalence.

It follows from Theorem C that there exists a well-defined restriction homomorphism $Res_H^G : Wh_G(X) \longrightarrow Wh_H(X)$. It should also be possible to give an algebraic description of Res_H^G when both $Wh_G(X)$ and $Wh_H(X) = Wh_H(esh_H(X))$ are given in their algebraic forms as direct sums of ordinary Whitehead groups of various discrete groups.

However, in many cases one may need further functorial properties of the restriction homomorphism. If $K<H<G$ one obtains the diagram

$$\begin{tikzcd} Wh_G(X) \arrow[r, "Res_H^G"] \arrow[dr, "Res_K^G"'] & Wh_H(X) \arrow[d, "Res_K^H"] \\ & Wh_K(X) \end{tikzcd}$$

in which all three homomorphism Res_H^G , Res_K^H and Res_K^G are well-defined. But I <u>cannot</u> see that it would follow from Theorem C that the diagram commutes. In other words, although Theorem C is good enough to give us a well-defined restriction homomorphism Res_H^G , whenever $H<G$, it is <u>not</u> good enough for establishing the natural associativity property

$$\mathrm{Res}_K^H \circ \mathrm{Res}_H^G = \mathrm{Res}_K^G$$

The results discussed in Sections 3 and 4 are also relevant for the solution of this problem.

3. The rôle of G-simplicial complexes

In this section G is a compact Lie group. We begin by recalling the definition of equivariant simplexes. (See [8, Definition 1.4] and [10, Section 3].) Let $H_0, \ldots, H_n$ be closed subgroups of G such that $H_0 \supset H_1 \supset \ldots \supset H_n$. Then the standard equivariant n-simplex of type $(H_0, \ldots, H_n)$ is defined as follows. Consider the G-space $\Delta_n \times G$, where G acts trivially on Δ_n and by left multiplication on G, and define a relation $\sim$ in $\Delta_n \times G$ as follows. We set

$$(x,g)\sim(x,g') \Leftrightarrow gH_m = g'H_m \in G/H_m\ , \text{ when } \ x \in \Delta_m - \Delta_{m-1}\ ,\ 0 \leqq m \leqq n\ .$$

Then $\sim$ is an equivalence relaion in $\Delta_n \times G$, and we define

$$\Delta_n(G;H_0, \ldots, H_n) = (\Delta_n \times G)/\sim\ .$$

The space $\Delta_n(G;H_0, \ldots, H_n)$ is given the quotient topology from the natural projection $p: \Delta_n \times G \to \Delta_n(G;H_0, \ldots, H_n)$, and we have an induced action of G on $\Delta_n(G;H_0, \ldots, H_n)$. The orbit space of this action is Δ_n, and every point in $\Delta_m - \Delta_{m-1} (0 \leqq m \leqq n)$ has isotropy type equal to (H_m). We call $\Delta_n(G;H_0, \ldots, H_n)$ the standard equivariant n-simplex of type $(H_0, \ldots, H_n)$.

Let X be a compact G-space. A finite equivariant triangulation of X consists of a finite (ordinary) simplicial complex K and a triangulation $t: X \to X/G$ of the orbit space, such that for each n-simplex s of K there exist closed subgroups $H_0, \ldots, H_n$ of G and a G-homeomorphism

$$\alpha : \Delta_n(G;H_0, \ldots, H_n) \to \pi^{-1}(t(s)) ,$$

which on the orbit space level induces a linear homeomorphism from Δ_n to $t(s)$.

If X is a compact G-space for which the orbit space X/G can be given a triangulation such that all points in any open simplex have the same G-isotropy type, then X can be given a finite equivariant triangulation, see [10, Theorem 5.5].

Definition. A finite G-simplicial complex X is a compact G-space X together with a finite equivariant triangulation of X.

A finite G-simplicial complex X is in particular a finite G-CW complex, see [10, Proposition 6.1].

Our main new result here is the following.

Theorem D. Let G be a compact Lie group. Then every finite G-simplicial complex X can be equivariantly imbedded in some representation space W as a subanalytic set.

For the notion of a subanalytic set see Hironaka [6] and [7], and Hardt [4].

Thus every finite G-simplicial complex can be given the structure of a subanalytic set. Furthermore, the proof of Theorem D is such that if $i:X \to W$ and $i':X \to W'$ are two G-imbeddings given by Theorem D, then the G-homeomorphism $i' \circ i^{-1}: i(X) \to i'(X)$ is a subanalytic isomorphism between subanalytic sets. Hence every finite G-simplicial complex can be given a canonical subanalytic set structure (up to subanalytic isomorphisms).

As a consequence of Theorem D one obtains the following result.

Corollary E. Let X be a finite G-simplicial complex, where G is a compact Lie group. For any closed subgroup H of G the H-space X can be given the structure of an H-simplicial complex, and this can be done in such a way that one obtains a well-defined simple H-homotopy type for X.

It should here be recalled that equivariant Whitehead torsion is not a topological invariant. Therefore only the fact that one succeeds in giving the H-space X itself the structure of a finite H-simplicial complex does not give one a well-defined simple H-homotopy type for the H-space X, because an arbitrary H-homeomorphism between two H-simplicial complexes need not be a simple H-homotopy equivalence. Thus Corollary E gives more information than the, interesting and pleasing, fact that the H-space X itself can be given a finite H-simplicial complex structure.

4. Compact smooth G-manifolds

In this section G denotes a compact Lie group. Let M be a compact smooth G-manifold. (By smooth we mean C^∞ .) It is known that M can be given the structure of a finite G-CW complex or in fact the structure of a finite G-simplicial complex. See Matumoto [13] and Illman [8], as well as [10]. In these works one uses the fact that the orbit space M/G can be triangulated in such a way that all points in any open simplex have the same isotropy type. Using a suitable lifting procedure it is then possible to prove that M admits the structure of a finite G-simplicial complex. It should be pointed out that one is here only using a topological triangulation of M/G , i.e., an arbitrary homeomorphism from a finite simplicial complex onto M/G , satisfying the additional property concerning the isotropy type in open simplexes.

In [14], Matumoto and Shiota use subanalytic triangulations of the orbit space M/G , and as a consequence of a certain uniqueness property for such triangulations one obtains a uniqueness property for the corresponding equivariant triangulations of M . In particular one obtains in this way a well-defined simple G-homotopy type for M .

In order to obtain a subanalytic structure on M/G one proceeds as follows. Given a compact smooth G-manifold M there exists a real analytic G-manifold M_a which is G-diffeomorphic to M , and any two such real analytic G-manifolds are real analytically G-isomorphic, see [14, Theorems 1.3 and 1.2]. Thus we may as well from now on assume that M is a compact real analytic G-manifold. Then there exists an equivariant real analytic imbedding of M into some representation space W of G, see [14, Theorem 1.1]. By classical invariant theory there exists a G-invariant polynomial map $p:W \to \mathbb{R}^n$, from W into some euclidean space $\mathbb{R}^n$, which induces a proper imbedding $\bar{p}:W/G \to \mathbb{R}^n$. Thus $p(W)=\bar{p}(W/G)$ is a closed semi-algebraic subset of $\mathbb{R}^n$. It now follows that one obtains a subanalytic set structure on M/G, and this structure is unique up to subanalytic homeomorphisms.

The important fact about subanalytic sets in this context is that there is a very good and well behaving theory of subanalytic triangulations of subanalytic sets see Hironaka [7] and Hardt [4]. In particular every subanalytic set has a subanalytic triangulation and any two such triangulations have a common subdivision. In the situation we are considering one obtains that M/G can be given a subanalytic triangulation in which all points in any open simplex have the same isotropy type. Thus, in this case, one obtains a commutative diagram

$$(\#) \qquad \begin{tikzcd} X \arrow[r, "\tilde{t}"] \arrow[d, "p_0"'] & M \arrow[d, "p"] \\ X/G=K \arrow[r, "t"] & M/G \end{tikzcd} \quad .$$

where K is a finite simplicial complex, t is a subanalytic homeomorphism (satisfying the additional property concerning isotropy types), and X is a finite G-simplicial complex and $\tilde{t}$ is a G-homeomorphism.

Now, by Theorem D and the discussion following it, every finite G-simplicial complex can be given a canonical structure of a subanalytic set. Thus it is natural to ask if we can take $\hat{t}$ to be a subanalytic G-homeomorphism. In fact, in order to have a very well working theory of equivariant triangulations it is desirable to obtain a diagram like (#) with $\hat{t}$ a subanalytic G-homeomorphism. We now state the following result.

Theorem F. Let M be a compact real analytic G-manifold, where G is a compact Lie group. Then there exists an equivariant subanalytic triangulation of M.

The conclusion in Theorem F means that there exist a finite G-simplicial complex X, equipped with its canoncial structure as a subanalytic set, and a subanalytic G-homeomorphism $\hat{t}:X \to M$, which in the orbit spaces induces an ordinary subanalytic triangulation $t:K \to M/G$, where $K=X/G$ is a finite simplicial complex.

Now let H be a closed subgroup of G. If $\hat{t}:X \to M$ is a G-equivariant subanalytic triangulation and we restrict the transformation group to H we obtain by Corollary E a finite H-simplicial complex $X|_H$, with a well-defined simple H-homotopy type, and the H-map $\hat{t}|_H:X|_H \to M|_H$ is an H-equivariant subanalytic triangulation. This shows the following:

If we define the simple G-homotopy type of M using the class of G-equivariant subanalytic triangulations of M (and likewise define the simple H-homotopy type of M by using the class of H-equivariant subanalytic triangulations of M) we obtain that the operation of restricting the transformation group G to the subgroup H takes the simple G-homotopy type of M to the simple

H-homotopy type of M.

Thus, the class of equivariant subanalytic triangulations of compact smooth G-manifolds (made into real analytic G-manifolds) is a good class of triangulations, and in particular they give a solution to Problem 4.1 in [16].

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